EXPONENTIAL SERIES

The sum of the series 1 + x + $\frac{x^2}{2!}+\frac{x^3}{3!}+.......\infty$ is denoted by the number e

Some important expansion:-

1. $e^x = 1 + x + \frac{x^2}{2!}+\frac{x^3}{3!}+.......\infty$

2. $e^{-x} = 1 - x + \frac{x^2}{2!}-\frac{x^3}{3!}+.......\infty$

3. $\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+...............\infty$

4. $\frac{e^x-e^{-x}}{2}=x+\frac{x^3}{3!}+\frac{x^5}{5!}+.....\infty$

5. $e = 1 + 1 + 1/2! + 1/3! + 1/4! + ...\infty = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{n=0}^{\infty}(1/n!)$ = 2.718 (approx.)

6. $e^{-1} = 1/2! - 1/3! + 1/4! - 1/5 + ...\infty = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \sum_{n=0}^{\infty}(1/n!)$

7. $\frac{e+e^{-1}}{2}=1+1/2!+1/4!+.......\infty$

8. $\frac{e-e^{-1}}{2}$ = 1 + 1/3! + 1/5! +…… ∞

9. $\lim_{n\to\infty}\left(1\pm\frac{1}{n}\right)^{nx}=e^{\pm x}$, for all values of x.

10. e is an irrational number.

11. If a be any positive number such that $\log_e a = m$, then $a = e^m$ and

$a^x = e^{mx} = 1 + \frac{mx}{1!}+\frac{m^2x^2}{2!}+\frac{m^3x^3}{3!}+......+\frac{m^rx^r}{r!}+.......\infty$

i.e., $a^x = 1 + (x/1!)(\log_e a) + (x^2/2!)(\log_e a)^2 +......+ (x^r/r!)(\log_e a)^r +......$

TO PROVE THAT e LIES BETWEEN 2 AND 3

We have, $e = 1 + \frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+....\infty$

$\Rightarrow e = 2 + \frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+....\infty$ $\qquad \Rightarrow e - 2 = \frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...\infty$

$\Rightarrow$ e – 2 = a positive number

$\Rightarrow e > 2$

We know $n! > 2^{n-1}$ for all n > 2

$\therefore \frac{1}{n!} < \frac{1}{2^{n-1}}$ for all n > 2

$\Rightarrow \frac{1}{3!} < \frac{1}{2^2}, \frac{1}{4!} < \frac{1}{2^3}, \frac{1}{5!} < \frac{1}{2^4} \ldots$

$\Rightarrow \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots < \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots$

$\Rightarrow 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \ldots < 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots$

$\Rightarrow 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots$

$$\Rightarrow e < 1 + \frac{1}{\left(1 - \frac{1}{2}\right)} \left[\because 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots = \frac{1}{\left(1 - \frac{1}{2}\right)} = 2 \right]$$

$\Rightarrow e < 1 + 2 = 3$

From, (i) and (ii), we get $2 < e < 3$. Hence, e lies between 2 and 3.

Note: It should be noted that

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{n=k}^{\infty} \frac{1}{(n-k)!} = e$$

$$\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots..\infty = e - 1$$

$$\sum_{n=2}^{\infty} \frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots..\infty = e - 2$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots..\infty = e - 1$$

$$\sum_{n=0}^{\infty} \frac{1}{(n+2)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots..\infty = e - 2$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots..\infty = e - 2$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\ldots.. = \frac{e + e^{1}}{2} = \sum_{n=1}^{\infty} \frac{1}{(2n-2)!}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)!} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\ldots.. = \frac{e - e^{1}}{2}$$

General Term

$$e^{ax} = 1 + \frac{(ax)}{1!} + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \ldots.. + \frac{(ax)^n}{n!} + \ldots..\infty.$$

Therefore T_{n+1} = General term in the expansion of $e^{ax} = \frac{(ax)^n}{n!}$ and, coefficient of x^n in e^{ax} $\frac{a^n}{n!}$

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots.. + \frac{x^n}{n!} + \ldots..\infty,$$

Therefore T_{n+1} = General term in the expansion of $e^x \frac{x^n}{n!}$ and, coefficient of x^n in $e^x = \frac{1}{n!}$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + (-1)^n \frac{x^n}{n!} + ...\infty$$

Therefore T_{n+1} = General term in the expansion of $e^{-x} (-1)^n \frac{x^n}{n!}$ and, coefficient of x^n in $e^{-x} = \frac{(-1)^n}{n!}$

EXAMPLES:

1. Find the coefficient of x^n in the expansion of e^{e^x}.

Sol. Let e^x = z. Then

$$e^{e^x} = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{(e^x)^k}{k!} = \sum_{k=0}^{\infty} \frac{e^{kx}}{k!} = \left(1 + \frac{e^x}{1!} + \frac{e^{2x}}{2!} + \frac{e^{3x}}{3!} + ...\infty\right)$$

$$= 1 + \frac{1}{1!}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) + \frac{1}{2!}\left(\sum_{n=0}^{\infty} \frac{(2x)^n}{n!}\right) + \frac{1}{3!}\left(\sum_{n=0}^{\infty} \frac{(3x)^n}{n!}\right) + ...\infty$$

Coefficient of x^n in $e^{e^x} = \frac{1}{1!}\left(\frac{1}{n!}\right) + \frac{1}{2!}\left(\frac{2^n}{n!}\right) + \frac{1}{3!}\left(\frac{3^n}{n!}\right)...\infty$

$$= \frac{1}{n!}\left(\frac{1}{1!} + \frac{2^n}{2!} + \frac{3^n}{3!} + ...\infty\right)$$

2. To prove $\dfrac{1 + \frac{2^2}{2!} + \frac{2^4}{3!} + \frac{2^6}{4!} + ...}{1 + \frac{1}{2!} + \frac{2}{3!} + \frac{2^2}{4!} + ...} = e^2 - 1.$

Sol. L.H.S. $= \dfrac{\frac{1}{2^2}\left[2^2 + \frac{2^4}{2!} + \frac{2^6}{3!} + \frac{2^8}{4!} + ...\right]}{\frac{1}{2^2}\left[2^2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + ...\right]} = \dfrac{-1 + 1 + \frac{4}{1!} + \frac{4^2}{2!} + \frac{4^3}{4!} + ...}{1 + 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^2}{3!}} = \dfrac{-1 + e^4}{1 + e^2} = \dfrac{(e^2 - 1)(e^2 + 1)}{e^2 + 1}$

$= e^2 - 1.$

2. Prove that $\left(1 + \frac{1}{1.2} + \frac{1}{1.2.3} + ...\right)\left(1 - \frac{1}{1.2} + \frac{1}{1.2.3} - .....\right) = \frac{(e-1)^2}{e}$

Sol. L.H.S. $= \left(1 + \frac{1}{2!} + \frac{1}{3!} + ...\right)\left(1 - \frac{1}{2!} + \frac{1}{3!} - ....\right) = \left(-1 + 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} ....\right).\left[1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)\right]$

$= (-1 + e') (1 - e^{-1}) = (-1 + e) (1 - 1/e) = (e - 1)2/e$

3. Find the value of $(1 + 3) \log_e 3 + \frac{(1 + 3^2)}{2!} (\log_e 3)^2 + ...$

Sol. Given expression

$$\left[1 + \log_e 3 + \frac{(\log_e 3)^2}{2!} + \frac{(\log_e 3)^3}{3!} + ...\right] + \left[1 + 3\log_e 3 + \frac{(3\log_e 3)^2}{2!} + \frac{(3\log_e 3)^3}{3!} + ...\right] - 2$$

$= e^{\log 3} + e^{3\log 3} - 2. = 3 + 3^3 - 2 = 28.$

4. Prove the following: $1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + ...\infty = 5e$.

Sol. n^{th} term of the given series is

$$T_n = \frac{n^3}{n!} = \frac{n^2}{(n-1)!} = \frac{(n^2-1)+1}{(n-1)!} = \frac{n+1}{(n-2)!} + \frac{1}{(n-1)!} = \frac{(n-2)+3}{(n-2)!} + \frac{1}{(n-1)!} = \frac{1}{(n-3)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}$$

Now we cannot put n = 1, 2 in first series So taking first two terms from the given equation and putting n=3 in all the three series

we get $1+\frac{2^3}{2!} + \left[1 + \frac{1}{1!} + \frac{1}{2!} + ...\right] + 3\left[\frac{1}{1!} + \frac{1}{2!} + ...\right] + \left[\frac{1}{2!} + \frac{1}{3!} + ...\right]$ = 1+4+e + 3(e–1)+e–2 = 5e

5. $\frac{2}{1!} + \frac{2+4}{2!} + \frac{2+4+6}{3!} + ... = 3e$.

Sol. $T_n = \frac{2+4+6+....+2n}{n!} = 2\left(\frac{\sum n}{n!}\right) = \frac{n+1}{(n-1)!} = \frac{1}{(n-2)!} + \frac{2}{(n-1)!}$ Take the first term from the series and put n = 2 in the two series and get Answer : 3e

6. $\frac{1.4}{0!} + \frac{2.5}{1!} + \frac{3.6}{2!} + \frac{4.7}{3!} + ... = 11e$.

Sol. (nth term of A.P. 1, 2, 3, …..).

$$T_n = \frac{(n^{th} \text{termofA.P}1,2,3,.....)(n^{th}\text{termofA.P.}4,5,6,......)}{(n^{th}\text{termofA.P.}0,1,2,....)!} = \frac{\{1+(n-1).1\}\{4+(n-1).1\}}{\{0+(n-1)1\}!}$$

$$= \frac{n(n+3)}{(n-1)!} = \frac{n^2+3n}{(n-1)!} = \frac{(n-1)(n-2)+(6n-2)}{(n-1)!} = \frac{(n-1)(n-2)+6(n-1)+4}{(n-1)!}$$ or

$$T_n = \frac{1}{(n-3)!} + \frac{6}{(n-2)!} + \frac{4}{(n-1)!}$$

Now taking the first two terms and putting n = 3, 4, 5, ….., and adding, the given series we get ans.11e

7. Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)!}$

Sol. $T_n = \frac{1}{(n-1)!} - \frac{1}{n!} + \frac{1}{(n+1)!}$ = e – (e – 1) + (e – 2) = e – 1.

8. Find the coefficient of x^n in the expansion of $\frac{e^{7x}+e^x}{e^{3x}}$

Sol. We have, $\frac{e^{7x}+e^x}{e^{3x}} = e^{4x} + e^{-2x} = \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{4^n}{n!} x^n + \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} x^n$.

Therefore Coefficient of x^n in $\frac{e^{7x}+e^x}{e^{3x}} = \frac{4^n}{n!} + \frac{(-1)^n 2^n}{n!} = \frac{2^n}{n!}\{2^n + (-1)^n\}$

LOGARITHMIC SERIES

As you have studied in earlier classes that $\log_a x$ is a number N such that $a^N = x$ i.e. $\log_a x = N \Leftrightarrow x = a^N$. Here a is known as the base of the logarithm. In this chapter we will take the number e as the base of the logarithm unless otherwise stated. Logarithms of numbers calculated to the base e are called Naperian logarithms or Natural logarithms and logarithms to the base 10 are know as Common logarithms. Now we will obtain an expansion for $\log_e (1 + x)$ as a series of powers of x which is valid only when $|x| < 1$.

Expansion of log, (1 + x): If $|x| < 1$, then $\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$

This series is known as the **Logarithmic series**

Note. Using summation notation $\sum$, we have $\log_e (1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

SOME IMPORTANT DEDUCTIONS FROM THE LOGARITHMIC SERIES

1. Replacing x by – x in the logarithmic series, we get: $\log_e (1 - x) = - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ....\infty$

   or $-\log_e (1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...\infty$

2. We have: $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...\infty$ and,

   $$-\log_e (1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...\infty$$

   Adding these two series, we get

   $$\log_e (1 + x) - \log_e (1 - x) = \log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\infty\right)$$

   Subtracting above series we get

   $$\log_e (1 + x) + \log_e (1 - x) = \log_e(1 - x^2) = -2\left(\frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + ...\infty\right)$$

3. The series expansion of $\log_e(1 + x)$ may fail to be valid if If $|x|$ is not less then 1. It can be proved that the logarithmic series is valid for x = 1 Putting x = 1 in the logarithmic series, we get

   $$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} .....\infty$$

4. When x = – 1, the logarithmic series does not have a sum. This is in conformity with the fact that log (1 –1) is not a finite quantity.

EXAMPLES:

1. Using the series for log 2, prove that the value of log 2 lies between 0.61 and 0.76.

Sol. We have:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots\infty$$

Putting x = 1 in this series, we get

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \ldots$$

$$= \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \ldots \geq \frac{37}{60} > 0.616$$

Also, $\log 2 = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \ldots$

$$= 1 - \frac{1}{6} - \frac{1}{20} - \frac{1}{42} - \ldots \leq 1 - \frac{1}{6} - \frac{1}{20} - \frac{1}{42} = \frac{319}{420} < 0.76$$

Hence, 0.61 < log 2 < 0.76.

2. Prove that

(i) $\left(\frac{1}{3} + \frac{1}{3.3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \ldots\ldots\infty\right) = \frac{1}{2}\log_e 2$

(ii) $\left(1 + \frac{1}{3.2^2} + \frac{1}{5.2^4} + \frac{1}{7.2^6} + \ldots.\infty\right) = \log_e 3.$

Sol. (i) We have : $\left(\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \ldots\ldots\infty\right) = \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\ldots\infty\right)$, where $x = \frac{1}{3}$

$$= \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\log_e\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = \frac{1}{2}\log_e 2.$$

(ii) We have $\left(1 + \frac{1}{3.2^2} + \frac{1}{5.2^4} + \frac{1}{7.2^6} + \ldots.\infty\right) = 2\left(\frac{1}{2} + \frac{1}{3.2^3} + \frac{1}{5.2^5} + \frac{1}{7.2^7} + \ldots.\infty\right)$

$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\ldots\infty\right)$, where x = ½ $= \log_e\left(\frac{1+x}{1-x}\right) = \log_e\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \log_e 3.$

2. Find the sum of the infinite series : $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \frac{1}{4.9} + \ldots\ldots\infty$

Sol. Let T_n be the n^{th} term of the given series. Then, $T_n = \frac{1}{n(2n+1)}$, n = 1, 2, 3, …..

Let $\frac{1}{n(2n+1)} = \frac{A}{n} + \frac{B}{(2n+1)}$

Or 1 = A(2n + 1) + B n

Comparing the coefficients of n and constant terms on the two sides of the above equation, we get 2A + B = 0, A = 1 and B = – 2. Substituting the values of A and B in (i), we obatin

$T_n = \frac{1}{n(2n+1)} = \frac{1}{n} - \frac{2}{(2n+1)}$, n = 1, 2, 3,……

Therefore Sum of the given series = $\sum_{n=1}^{\infty} T_n = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{2}{2n+1}\right)$

$= \left[\left(1 - \frac{2}{3} + \left(\frac{1}{2} + \frac{2}{5}\right) + \left(\frac{1}{3} - \frac{2}{7}\right) + ...\infty\right)\right] = \left(1 + \frac{1}{2} + \frac{1}{3} + ...\infty\right) - 2\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ...\infty\right)$

$= 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + ...\infty = 2 - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...\infty\right) = 2 - \log_e 2.$

3. Prove that $\log_e x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + ...$

[**Hint :** $\log_e x = \log_e \{1 + (x-1)\} = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - .....$]

4. Prove that : $\frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + ...\infty = (2\log_e 2) - 1$

Hint : $\left[\text{LHS} = \left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) + ...\infty = 1 + 2\left[\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...\right) + ...1\right] = 1 + 2\log_e 2 - 2\right]$

5. Prove that : $\left[\frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{4}\left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \frac{1}{6}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + ...\infty\right] = \log_e \sqrt{2}$

$\left[\text{HINT : Given series} = \frac{1}{2}\left[\left\{\frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 + ...\right\} + \left\{\frac{1}{3} - \frac{1}{2}\left(\frac{1}{3}\right)^2 + \frac{1}{3}\left(\frac{1}{3}\right)^3 - ......\infty\right\}\right]\right]$

$= \left[\frac{1}{2}\left\{\log_e\left(1 + \frac{1}{2}\right) + \log\left(1 + \frac{1}{3}\right)\right\}\right]$

6. Prove that $\log(1 + 3x + 2x^2) = 3x - \frac{5}{2}x^2 + \frac{9}{3}x^3 - \frac{17}{4}x^4 + ...$

Sol. $\log\{1 + 3x + 2x^2) = \log\{(1-x)(1+2x)\} = \log(1+x) + \log(1+2x)$

$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...\infty\right) + \left(2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + ...\infty\right)$

$= 3x - x^2\left(\frac{1}{2} + \frac{2^2}{2}\right) + x^3\left(\frac{1}{3} + \frac{2^3}{3}\right) - x^4\left(\frac{1}{4} + \frac{2^4}{4}\right) + ... = 3x - \frac{5}{x}x^2 + \frac{9}{3}x^3 - \frac{17}{4}x^4 + ...$

7. Find the coefficient of x^n in the expansion of $\log_e(1 + 3x + 2x^2)$

Sol. We have: $\log_e(1 + 3x + 2x^2) = \log_e\{(1+x)(1+2x)\} = \log_e(1+x) + \log_e(1+2x)$

$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - ..... + (-1)^{n-1}\frac{x^n}{n} + ...\right) + \left(2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - ..... + (-1)^{n-1}\frac{(2x)^n}{n} + ...\right)$

Therefore Coefficient of x^n in $\log_e(1 + 3x + 2x^2) = \frac{(-1)^{n-1}}{n} + (-1)^{n-1}\frac{2^n}{n} = \frac{(-1)^{n-1}}{n}(1 + 2^n)$